THEORIES OF MAXWELLIAN DESIGN

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Some historical background. Anyone who has attempted to discover within the collected works of (say) Newton, Lagrange or Hamilton the equations that today celebrate the names and accomplishment of those great figures will not be surprised to learn that it requires patient searching to discover "Maxwell's equations" within *The Scientific Papers of James Clerk Maxwell*.[1] New babies are invariably dirty, and must be washed of extraneous material (recently so important to their prenatal lives) before we can appreciate how cute they are. And physics new a century or more ago tends to the modern eye (so unlike old music to the modern ear) to seem notationally—and sometimes also conceptually—quite obscure.

Maxwell's own notational conventions—he exhausted the alphabetical resources of several languages—were so cumbersome (and so consistently unlike their modern counterparts) that it seems wonderous that he was able to accomplish anything, or to remember from one day to the next what his symbols stood for. But in his "A Dynamical Theory of the Electromagnetic Field"[2] the evidence of the text suggests clearly that he was at special pains underscore the essence of his accomplishment. Equations in that paper are numbered consecutively, from beginning to end, except for eight sets of equations displayed in his "PART III. GENERAL EQUATIONS FOR THE ELECTROMAGNETIC FIELD," which he identifies by letter. Those equations—"Maxwell's equations," as they came first into the world—are so seldom seen by modern eyes[3] that I reproduce

[1] Edited by W. D. Niven, and originally published in 1890, but now available (two volumes bound as one) from Dover Pulications.

[2] Royal Society Transactions, 1864. Maxwell was then 33 years old.

[3] See, however, pp. 255–267 of R. Tricker's *The Contributions of Faraday & Maxwell to Electrical Science* (1966). Curiously, Maxwell—who was not at all averse to the use of indexed variables (they are abundant in his papers relating to the kinetic theory of gases)—opted to make no use of that notational device in his electrodynamical writing.

them below:

$$\left.\begin{aligned} p' &= p + \tfrac{df}{dt} \\ q' &= q + \tfrac{dg}{dt} \\ r' &= r + \tfrac{dh}{dt} \end{aligned}\right\} \textit{total currents} \qquad (A)$$

$$\left.\begin{aligned} \mu\alpha &= \tfrac{dH}{dy} - \tfrac{dG}{dz} \\ \mu\beta &= \tfrac{dF}{dz} - \tfrac{dH}{dx} \\ \mu\gamma &= \tfrac{dG}{dx} - \tfrac{dF}{dy} \end{aligned}\right\} eq^S \textit{ of magnetic force} \qquad (B)$$

$$\left.\begin{aligned} \tfrac{d\gamma}{dy} - \tfrac{d\beta}{dz} &= 4\pi p' \\ \tfrac{d\alpha}{dz} - \tfrac{d\gamma}{dx} &= 4\pi q' \\ \tfrac{d\beta}{dx} - \tfrac{d\alpha}{dy} &= 4\pi r' \end{aligned}\right\} eq^S \textit{ of electric currents} \qquad (C)$$

$$\left.\begin{aligned} P &= \mu\big(\gamma\tfrac{dy}{dt} - \beta\tfrac{dz}{dt}\big) - \tfrac{dF}{dt} - \tfrac{d\psi}{dx} \\ Q &= \mu\big(\alpha\tfrac{dz}{dt} - \gamma\tfrac{dx}{dt}\big) - \tfrac{dG}{dt} - \tfrac{d\psi}{dy} \\ R &= \mu\big(\beta\tfrac{dx}{dt} - \alpha\tfrac{dy}{dt}\big) - \tfrac{dH}{dt} - \tfrac{d\psi}{dz} \end{aligned}\right\} eq^S \textit{ of electromotive force} \qquad (D)$$

$$\left.\begin{aligned} P &= kf \\ Q &= kg \\ R &= kh \end{aligned}\right\} eq^S \textit{ of electric elasticity} \qquad (E)$$

$$\left.\begin{aligned} P &= -\rho p \\ Q &= -\rho q \\ R &= -\rho r \end{aligned}\right\} eq^S \textit{ of electric resistance} \qquad (F)$$

$$e + \tfrac{df}{dx} + \tfrac{dg}{dy} + \tfrac{dh}{dz} = 0 \quad \textit{equation of free elecricity} \qquad (G)$$

$$\tfrac{de}{dt} + \tfrac{dp}{dx} + \tfrac{dq}{dy} + \tfrac{dr}{dz} = 0 \quad \textit{equation of continuity} \qquad (H)$$

These are twenty equations in as many variables:

components of *electromagnetic momentum*............	F	G	H
components of *magnetic intensity*....................	α	β	γ
components of *electromotive force*....................	P	Q	R
components of *current due to true conduction*.........	p	q	r
components of *electric displacement*...................	f	g	h
components of *total current*..........................	p'	q'	r'
quantity of free electricity............................	e		
electric potential......................................	ψ		

The symbols μ, k and ρ refer not to "variables" but to adjustable material parameters.

Ten years were to elapse before this material found its way into print again. Turning the pages of *A Treatise on Electricity & Magnetism* (1874) one finds that Maxwell had, during the interval, been motivated to adjust (slightly) his notation, to undertake some reformulation (by occasionally inserting one equation into another), and to present his equations in altered sequence. The fundamental equations—again identified by letter—are scattered throughout more than twenty pages of text spanning two chapters, and are now more numerous (and involve more variables) than before:

$$\left.\begin{aligned} a &= \tfrac{dH}{dy} - \tfrac{dG}{dz} \\ b &= \tfrac{dF}{dz} - \tfrac{dH}{dx} \\ c &= \tfrac{dG}{dx} - \tfrac{dF}{dy} \end{aligned}\right\} \qquad (B \to A)$$

$$\left.\begin{aligned} P &= \left(c\tfrac{dy}{dt} - b\tfrac{dz}{dt}\right) - \tfrac{dF}{dt} - \tfrac{d\psi}{dx} \\ Q &= \left(a\tfrac{dz}{dt} - c\tfrac{dx}{dt}\right) - \tfrac{dG}{dt} - \tfrac{d\psi}{dy} \\ R &= \left(b\tfrac{dx}{dt} - a\tfrac{dy}{dt}\right) - \tfrac{dH}{dt} - \tfrac{d\psi}{dz} \end{aligned}\right\} \textit{eq}^{\mathrm{S}} \textit{ of electromagnetic intensity} \qquad (D \to B)$$

$$\left.\begin{aligned} X &= vc - wb + eP - m\tfrac{d\Omega}{dx} \\ Y &= wa - uc + eQ - m\tfrac{d\Omega}{dy} \\ Z &= ub - va + eR - m\tfrac{d\Omega}{dz} \end{aligned}\right\} \textit{eq}^{\mathrm{S}} \textit{ of electromagnetic force} \qquad (C)$$

$$\left.\begin{aligned} a &= \alpha + 4\pi A \\ b &= \beta + 4\pi B \\ c &= \gamma + 4\pi C \end{aligned}\right\} \textit{eq}^{\mathrm{S}} \textit{ of magnetization} \qquad (D)$$

$$\left.\begin{aligned} 4\pi u &= \tfrac{d\gamma}{dy} - \tfrac{d\beta}{dz} \\ 4\pi v &= \tfrac{d\alpha}{dy} - \tfrac{d\beta}{dz} \\ 4\pi w &= \tfrac{d\beta}{dx} - \tfrac{d\alpha}{dy} \end{aligned}\right\} \textit{eq}^{\mathrm{S}} \textit{ of electric currents} \qquad (C \to E)$$

$$\left.\begin{aligned} f &= \tfrac{1}{4\pi}KP \\ g &= \tfrac{1}{4\pi}KQ \\ h &= \tfrac{1}{4\pi}KR \end{aligned}\right\} \textit{eq}^{\mathrm{S}} \textit{ of electric displacement} \qquad (E \to F)$$

$$\left.\begin{aligned} p &= CP \\ q &= CQ \\ r &= CR \end{aligned}\right\} \textit{eq}^{\mathrm{S}} \textit{ of conductivity (Ohm's law)} \qquad (F \to G)$$

$$\left.\begin{aligned} u &= p + \tfrac{df}{dt} \\ v &= q + \tfrac{dg}{dt} \\ w &= r + \tfrac{dh}{dt} \end{aligned}\right\} \textit{eq}^{\mathrm{S}} \textit{ of true currents} \qquad (A \to H)$$

$$\rho = \tfrac{df}{dx} + \tfrac{dg}{dy} + \tfrac{dh}{dz} \qquad (G \to J)$$

$$\left.\begin{aligned} a &= \mu\alpha \\ b &= \mu\beta \\ c &= \mu\gamma \end{aligned}\right\} \textit{eq}^{\mathrm{S}} \textit{ of induced magnetization} \qquad (L)$$

The "equation of continuity" no longer appears on Maxwell's list, presumably because it is a corollary of the equations that do appear; on those same grounds I have omitted Maxwell's equations (I) and (K).

The final two articles of Maxwell's Chapter IX appear under the section head "*Quaternion Expressions for the Electromagnetic Equations*." Maxwell remarks that while "we have not scrupled to introduce the idea of a vector when [only seldom] it was necessary to do so," he has "endeavoured to avoid any process demanding from the reader a knowledge of the Calculus of Quaternions." Concerning the "quaternions" to which he alludes...

On 16 October 1843 Hamilton hit upon the essentials[4] of the "quaternions" which were to command his attention until his death (in 1865). The cause (for so it came to be viewed by a vocal cadre) was taken up by—among others—Peter Guthrie Tait, who had been a classmate of Maxwell, and remained his lifelong friend. One gains the impression that Maxwell himself, though not insensitive to the charm of certain quaternionic developments, was reluctant to subject physics to the rigid constraints of *any* doctrinaire formalism, or to express himself in a language which readers might not comprehend—particularly since the formalism in this instance permitted him to do nothing he could not equally well do by more standard means; one gains the impression that it was more from friendship than from scientific conviction[5] that Maxwell inserted into his *Treatise* the §618 and §619 which, though they run in total to scarcely two pages, were destined to engage the attention of Heaviside and Gibbs, and thus to change forever the face of physics... but I run ahead of myself.

In §618 Maxwell supplies a named list of the vectors (and a shorter list of the scalars) of which he has made implicit use. He then (§619) observes that if the terms "vector" and "scalar" are understood to refer to the *vector/scalar parts of quaternions*,[6] and if ∇ is understood to be quaternion-valued, then the

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[4] The expression $Q = q^0 + iq^1 + jq^2 + kq^3$ becomes a "quaternion" when the objects $\{i, j, k\}$ are required to satisfy what Hamilton called the "multiplication assumptions"

$$ii = jj = kk = -1$$
$$ij = -ji = k \qquad jk = -kj = i \qquad ki = -ik = j$$

which are collectively equivalent to the statement

$$\begin{aligned} PQ = \big\{p^0q^0 - p^1q^1 - p^2q^2 - p^3q^3\big\} &+ i\,\big\{\big(p^0q^1 + p^1q^0\big) + \big(p^2q^3 - p^3q^2\big)\big\} \\ &+ j\,\big\{\big(p^0q^2 + p^2q^0\big) + \big(p^3q^1 - p^1q^3\big)\big\} \\ &+ k\big\{\big(p^0q^3 + p^3q^0\big) + \big(p^1q^2 - p^2q^1\big)\big\} \end{aligned}$$

[5] Maxwell had promised Tait that he "would sow 4nion seed at Cambridge," but appears to have been a somewhat reluctant farmer.

[6] Maxwell writes

$$\begin{aligned} S.Q &\equiv \text{scalar part of } Q \equiv q^0 \\ V.Q &\equiv \text{vector part of } Q \equiv iq^1 + jq^2 + kq^3 \end{aligned}$$

field equations can be notated

$$
\begin{aligned}
\mathfrak{B} &= V.\nabla\mathfrak{A} && (A)\\
\mathfrak{E} &= V.\mathfrak{G}\mathfrak{B} - \dot{\mathfrak{A}} - \nabla\Psi && (B)\\
\mathfrak{F} &= V.\mathfrak{C}\mathfrak{B} + e\mathfrak{E} - m\nabla\Omega && (C)\\
\mathfrak{B} &= \mathfrak{H} + 4\pi\mathfrak{J} && (D)\\
4\pi\mathfrak{C} &= V.\nabla\mathfrak{H} && (E)\\
\mathfrak{D} &= \tfrac{1}{4\pi}K\mathfrak{E} && (F)\\
\mathfrak{K} &= C\mathfrak{E} && (G)\\
\mathfrak{C} &= \mathfrak{K} + \dot{\mathfrak{D}} && (H)\\
e &= S.\nabla\mathfrak{D} && (J)\\
m &= S.\nabla\mathfrak{C} &&\\
\mathfrak{B} &= \mu\mathfrak{H} && (L)\\
\mathfrak{H} &= -\nabla\Omega &&
\end{aligned}
$$

Thus ends Maxwell's Chapter IX: "General Equations of the Electromagnetic Field." The Gothic characters—stripped, however, of their quaternionic burden—recur in his very interesting Chapter X ("Dimensions of Elecric Units"), but then disappear altogether. Maxwell himself, as I have already remarked, appears to have been not much interested in the quaternionic expression of his theory.[7]

There is—quite apart from its quaternionic costume—much that is curious about the preceding formulation of the "general equations of electrodynamics."

[7] I cite in evidence the high density of typographic errors that Maxwell allowed to survive in the text; I work, however, from the posthumous 3^{rd} edition (1891), so the errors may have been introduced by someone else (the editor was J. J. Thompson). Patterns evident in the errors suggest that Maxwell (or perhaps his typesetter) was not comfortable with Gothic typefaces. It would be interesting to know whether boldface characters (also the symbol ∂) were unavailable to him; neither seems to occur anywhere in his published work.

The drift of Maxwell's famous essay "On the mathematical classification of physical quantities" (1870) [see p. 257 of his *Scientific Papers, Volume II*] suggests that Maxwell considered quaternion-based notions to be merely a step in the right direction; that he stood poised to embrace the tensor/spinor analysis that lay still several decades into the future.

It was, by the way, near the end of the essay to which I have just referred that Maxwell introduced the now-familiar terms "gradient," "convergence" (negative of the "divergence;" the latter term was used by Clifford in 1878, but is usually attributed to Heaviside (1883)) and "curl." Concerning the latter, he remarks that he had considered "rotation," "whirl" and "twirl," only to reject them because they "connote motion," and that he had rejected "twist" because it brings to mind "a helical or screw structure which is not of the nature of a vector at all."

I shall postpone commentary, however, until we have acquired access to more modern notational devices; as it happens, it was none other than the equations now before us that *inspired the invention* of the familiar devices I have in mind.

J. Willard Gibbs was a recently appointed professor of mathematical physics at Yale, whose developed interest in thermodynamics had already borne ripe fruit and earned for him the respect of Maxwell (eight years his senior), when in the mid-1870's he took up study of the *Treatise*. There, in §619, he made his "first acquaintance with quaternions," which motivated him to look into Hamilton's (posthumous) *Elements of Quaternions* (1866) and Tait's *Elementary Treatise on Quaternions* (1867). It became soon apparent to him that quaternions carried a lot of excess baggage as they went about the simple work they were being asked to do; only 3-element "vectorial quaternions" were encountered in (electrodynamical) applications, so if one wrote

$$\begin{aligned}\mathbf{P} &= \mathbf{i}p^1 + \mathbf{j}p^2 + \mathbf{k}p^3\\ \mathbf{Q} &= \mathbf{i}q^1 + \mathbf{j}q^2 + \mathbf{k}q^3\end{aligned}$$

and—drawing inspiration from the quaternionic multiplication formula (see again footnote 4)—defined two kinds of "vector product"

$$\begin{aligned}\mathbf{P}\cdot\mathbf{Q} &= p^1q^1 + p^2q^2 + p^3q^3\\ &= \text{number-valued “direct product”}\\ \mathbf{P}\times\mathbf{Q} &= \mathbf{i}(p^2q^3 - p^3q^2) + \mathbf{j}(p^3q^1 - p^1q^3) + \mathbf{k}(p^1q^2 - p^2q^1)\\ &= \text{vector-valued “skew product”}\end{aligned}$$

then all the remainder of the quaternionic apparatus could be abandoned.[8] Gibbs worked out the self-consistent implications of such a procedure, and from 1879 onward regularly taught a ninety-lecture course in "vector analysis and its applications." In 1884 he "printed but did not publish" his class notes, and distributed copies of the resulting *Elements of Vector Analysis* to more than one hundred leading scientists of the day, among them Rayleigh, Stokes, Kelvin, Cayley, Tait, Helmholtz, Claussius, Kirchhoff, Lorentz... and Heaviside.[9]

[8] Here casually abandoned was a capability which for Hamilton was central, namely any possibility of assigning meaning to *ratios*; P/Q has meaning if P and Q are quaternions with $q^0q^0 + q^1q^1 + q^2q^2 + q^3q^3 \neq 0$, but $\mathbf{P}/\mathbf{Q}$ is in all cases meaningless.

[9] Concerning the subsequent history of those notes: In 1899 Edwin Bidwell Wilson, having just received a BA in mathematics from Harvard, went to Yale for graduate study. The Harvard faculty was very "quaternionic" in those days, and Wilson, a young expert in the field, was initially not much pleased when, to meet a load requirement, he found himself obligated to take (redundantly, as he imagined) the "Vector Analysis" taught by a Professor Gibbs in the physics department. He found the course very easy, and in 1901 took his PhD. A Professor Morris was at the time editor of the Yale Bicentennial Series, and

Oliver Heaviside, born in 1850, was nineteen years younger than Maxwell, and eleven years younger than Gibbs. Forced to abandon formal education at the age of sixteen, he in 1868 obtained employment as a telegraph operator (probably through influence of his uncle, Sir Charles Wheatstone, and rather like Thomas Edison, his contemporary on the other side the Atlantic), and developed an interest in electrical theory. In 1874, with several publications already to his credit and a copy of Maxwell's *Treatise* in hand, he retired to the home of his impoverished parents in order to devote himself fulltime to electromagnetic research. Maxwell's §619 sent him to Tait's monograph, but he soon found that the theory of quaternions pertained most usefully to physical problems if one retained only the scalars, vectors and some simple vector algebra, but abandoned the quaternions themselves. He later wrote, in a review of Wilson's *Vector Analysis*, that

> *"Up to 1888 I had imagined that I was the only one doing vectorial work on positive physical principles; but then I received a copy of Prof. Gibbs's Vector Analysis (unpublished, 1881–1884). This was a sort of condensed synbopsis of a treatise. Though different in appearance, it was essentially the same vectorial algebra and analysis to which I had been led."*

Heaviside's own vectorial innovations were introduced on an as-needed basis into his many electrical papers, and were summarized for the first time in the introduction to a paper (Philosophical Magazine, 1885) which bore the title "On the electromagnetic wave surface."

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desirous that Gibbs' notes should be prepared for inclusion in the series. Gibbs himself, however, was too busy with his statistical mechanics to attend to such a chore, which fell therefore to Wilson, who was given free reign (Gibbs could find time to read neither the manuscript or the proofs) and in late 1901, at the age of twenty-two, completed *Vector Analysis: A Textbook for the Use of Students of Mathematics and Physics*. The book—the first of the genre, and still in some ways one of the best—established the organizational and notational conventions which remain standard to the field. Wilson himself went on to write a calculus that for decades led the field, engaged Hilbert in published debate concerning the foundations of geometry, and contributed to the early development of the theory of relativity. He was appointed head of the physics department of MIT in 1917, but five years later moved to the Harvard School of Public Health, where he was Professor of Vital Statistics and acquired a major reputation as a statistician. At Harvard he exerted a major influence during the 1930's upon the young Paul Samuelson (see the introduction to the most recent edition of *Foundations of Economic Analysis*). For fifty years (!) he served as editor of the Proceedings of the National Acadamy of Sciences. It was, however, a *different* E. B. Wilson (Edgar Bright Wilson) who—also and simultaneously at Harvard, but in the chemistry department—co-authored *Introduction to Quantum Mechanics* (1935) with Linus Pauling, and became the father of Nobel-laurette Kenneth G. Wilson. Until recently I have attributed *all* of those accomplishments to the same amazing fellow.

Heaviside's difficult, and in many respects eccentric, papers were at first not well received. But Hertz' discovery of electromagnetic waves in 1887 did much to remove lingering resistance to Maxwellian electrodynamics, and appreciative reviews of Heaviside's *Electrical Papers* (1892) and *Electromagnetic Theory* (1893) sparked expanding interest not only in Heaviside's physics but also in his mathematical methodology. Gibbs remained at the time much less well known to the European physics community, so it was mainly to Heaviside that the continental physicists of the day owed their vectorial educations.[10] There were pockets of resistance—Tait held Gibbs' pamphlet to be "a sort of hermaphrodite monster, compounded of the notations of Hamilton and of Grassmann," while Kelvin, a lifelong friend and collaborator of Tait but never a friend of Tait's beloved quaternions, wrote in 1896 to FitzGerald that "...'vector' is a useless survival, or offshoot, from quaternions, and has never been of the slightest use to any creature..." —but more typical was the response of August Föppl, whose *Einfuhrung in die Maxwell'sche Theorie der Elektricität* appeared in 1894. This influential text gave the first careful account of Maxwellian electrodynamics to appear in German;[11] the first three chapters provided readers with a systematic introduction to vector analysis in the tradition of Heaviside, concerning whom the author had this to say:

> *"In the presentation of vector techniques and on many other points, I followed the pattern set by O. Heaviside... The works of this author have in general influenced my presentation more than those of any other physicist with the obvious exception of Maxwell himself. I consider Heaviside to be the most eminent successor to Maxwell in regard to theoretical developments..."*

Throughout the 1890's the vectorial methods promoted by Gibbs and Heaviside were on the ascendancy and quaternions in steady decline,[12] so that by 1904 Lorentz could assume he would be understood when he remarked, near the beginning of the paper[13] which announced the discovery of (what we today

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[10] It would appear to be from Heaviside that Wilson borrowed the convention of using boldface type to denote vectors; Heaviside took exception both to the Greek employed by Hamilton, Tait and (in conscious deference to them) Gibbs, and to the Gothic which had been adopted by Maxwell, and in a paper of 1886 remarks that "...I found salvation in **Clarendons**, and introduced the use of [such] type for vectors, and have found it thoroughly suitable."

[11] It was from this text that the young Einstein reportedly gained his basic command of electrodynamics. The text—known first as Föppl, then as Föpple & Abraham, then as Abraham & Becker, then as Becker & Sauter—had gone through sixteen editions by 1957, and the unnumbered edition on my own bookshelf is dated 1964.

[12] They were destined to experience a reincarnation thirty years later as the "Pauli matrices" fundamental to the quantum theory of spin.

[13] "Electromagnetic phenomena in a system moving with any velocity less than that of light," which is reproduced in the Dover reprint collection entitled *The Principle of Relativity*.

call) the "Lorentz-covariance of the electromagnetic field equations," that what he called "the fundamental equations of the theory of electrons" can be written

$$\begin{aligned} \operatorname{div}\mathbf{D} &= \rho \\ \operatorname{div}\mathbf{H} &= 0 \\ \operatorname{curl}\mathbf{H} &= \tfrac{1}{c}\big(\tfrac{\partial\mathbf{D}}{\partial t} + \rho\mathbf{v}\big) \\ \operatorname{curl}\mathbf{D} &= -\tfrac{1}{c}\tfrac{\partial\mathbf{H}}{\partial t} \\ \mathbf{F} &= \mathbf{D} + \tfrac{1}{c}\mathbf{v}\times\mathbf{H} \end{aligned}$$

It is instructive to compare these with the equations that result when Maxwell's equations (A–L) are subjected to quaternion $\mapsto$ vector notational conversion:

$$\begin{aligned} \mathbf{B} &= \operatorname{curl}\mathbf{A} && (A^*) \\ \mathbf{E} &= \mathbf{v}\times\mathbf{B} - \tfrac{\partial\mathbf{A}}{\partial t} - \nabla\Psi && (B^*) \\ \mathbf{F} &= \mathbf{I}\times\mathbf{B} + \rho\mathbf{E} - \rho_m\nabla\Omega && (C^*) \\ \mathbf{B} &= \mathbf{H} + 4\pi\mathbf{M} && (D^*) \\ 4\pi\mathbf{I} &= \operatorname{curl}\mathbf{H} && (E^*) \\ \mathbf{D} &= \tfrac{1}{4\pi}\epsilon\mathbf{E} && (F^*) \\ \mathbf{J} &= \sigma\mathbf{E} && (G^*) \\ \mathbf{I} &= \mathbf{J} + \tfrac{\partial\mathbf{D}}{\partial t} && (H^*) \\ \rho &= \operatorname{div}\mathbf{J} && (J^*) \\ \rho_m &= \operatorname{div}\mathbf{I} && \\ \mathbf{B} &= \mu\mathbf{H} && (L^*) \\ \mathbf{H} &= -\nabla\Omega && \end{aligned}$$

We stand in need of a modern account of *Maxwell's own* conception of the theory we attribute to him.[14] But even in the absence of such an essay, it is

[14] Maxwell drew heavily upon the theory of fluids and elastic media, so we find semi-intelligible the fact that he made no effort to separate "properties of the electromagnetic field" from "properties of ponderable matter." But is the curious sequence in which he presents his equations (Ohm's law (G) preceeds Gauss' law (J)) an expository accident or a reflection of his sense of the logic of the situation? His (A) entails $\nabla\cdot\mathbf{B} = 0$, but from that he refuses to draw the conclusion that "magnetic monopoles do not exist;" indeed, he makes explicit provision for the possibility that "magnetic matter" might in fact be found to exist. Maxwell's (B) contains a term $\mathbf{v}\times\mathbf{B}$ in which we have learned to read an allusion to "transformation theory," and only when that term is dropped do (A) and (B) jointly entail $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$. These are only the most obvious of the points that the essay I have in mind would undertake to clarify. I gather from a review (AJP **66**, 92 (1998)) that first steps in the direction I have indicated have been taken by T. K. Simpson in *Maxwell on the Electromagnetic Field: A Guided Study* (1997).

clear that, however great may have been the notational transmogrification that took place between 1874 and 1904, the conceptual adjustment was even greater and more profound.

The conceptual development to which I have just alluded drew impetus from three principal sources. I am thinking here first of all of Heinrich Hertz' experimental effort—begun in 1879 and brought to a successful conclusion in 1888—to establish the physical existence of electromagnetic radiation. This work was pursuant to Maxwell's *Chapter XX: Electromagnetic Theory of Light*, and stimulated the development (by Joseph Larmor and others) of the theory of radiative processes. A second major development—associated with the names of Albert Michelson and Edward Morley, and with the null results achieved by experiments performed during the 1880's—drew inspiration from the words with which Maxwell (§866: "The idea of a medium cannot be got rid of") brought the *Treatise* to a close:

> *"We have seen that the mathematical expressions for electrodynamic action led, in the mind of Gauss, to the conviction that a theory of the propagation of electric action in time would be found to be the very keystone of electrodynamics. Now we are unable to conceive of propagation in time, except either as the flight of a material substance through space, or as the propagation of a condition of motion or stress in a medium already existing in space. In the theory of Neumann, the mathematical conception called Potential, which we are unable to conceive as a material substance, is supposed to be projected from one particle to another, in a manner which is quite independent of a medium, and which, as Neumann has himself pointed out, is extremely different from that of the propagation of light. In the theories of Riemann and Betti it would appear that the action is supposed to be propagated in a manner somewhat more similar to that of light.*
>
> *But in all of these theories the question naturally occurs:—If something is transmitted from one particle to another at a distance, what is its condition after it has left the one particle and before it has reached the other? If this something is the potential energy of the two particles, as in Neumann's theory, how are we to conceive this energy as existing in a point of space, coinciding neither with the one particle nor with the other? In fact, whenever energy is transmitted from one body to another in time, there must be a medium or substance in which the energy exists after it leaves one body and before it reaches the other, for energy, as Torricelli remarked, 'is a quintessence of so subtile a nature that it cannot be contained in any vessel except the inmost substance of material things.' Hence all these theories lead to the conception of a medium in which the propagation takes place, and if we admit this medium as an hypothesis, I think it ought to occupy a prominent place in our investigations, and that we ought to endeavor to construct a mental*

> *representation of all the details of its action, and this has been my constant aim in this treatise."*

The last of the developments I have in mind—and the only one not pursuant to ideas put forward in the *Treatise*—was the discovery (by J. J. Thompson, in 1897) of the electron, the theoretical ramifications of which were almost instantaneous, and enormous.[15]

These developments served—each in its own way—to inspire an effort to construct a variant of Maxwell's theory which is stripped of all distracting reference to properties of "ponderable matter," a theory in which the only players are electromagnetic fields, point-like sources (electrons) and (or so it was imagined) the elusive "æther" (imponderable matter?). The way out of the woods, however obvious it may seem in retrospect, was at the time not at all obvious, for it entailed fresh thought of a deeply physical sort. Many people—Lorentz prominent among them—contributed to this work; when one speaks the "Maxwell-Lorentz equations" one is, in effect, bracketing an intense effort that spanned more than two decades.

Einstein, who was born in 1879 (the year of Maxwell's death), came of age when this work was freshly complete. In the second, or "Electrodynamical Part," of his celebrated relativity paper[16] Einstein refers without citation to what he calls the "Maxwell-Hertz equations," which ("when convection currents are taken into account;" see his §9) he takes to read

$$\begin{aligned}
\tfrac{1}{c}\Big\{\tfrac{\partial X}{\partial t} + u_x\rho\Big\} &= \tfrac{\partial N}{\partial y} - \tfrac{\partial M}{\partial z}\\
\tfrac{1}{c}\Big\{\tfrac{\partial Y}{\partial t} + u_y\rho\Big\} &= \tfrac{\partial L}{\partial z} - \tfrac{\partial N}{\partial x}\\
\tfrac{1}{c}\Big\{\tfrac{\partial Z}{\partial t} + u_z\rho\Big\} &= \tfrac{\partial M}{\partial x} - \tfrac{\partial L}{\partial y}\\
\tfrac{1}{c}\tfrac{\partial L}{\partial t} &= \tfrac{\partial Y}{\partial z} - \tfrac{\partial Z}{\partial y}\\
\tfrac{1}{c}\tfrac{\partial M}{\partial t} &= \tfrac{\partial Z}{\partial x} - \tfrac{\partial X}{\partial z}\\
\tfrac{1}{c}\tfrac{\partial N}{\partial t} &= \tfrac{\partial X}{\partial y} - \tfrac{\partial Y}{\partial x}\\
\rho &= \tfrac{\partial X}{\partial x} + \tfrac{\partial Y}{\partial y} + \tfrac{\partial Z}{\partial z}
\end{aligned}$$

Einstein remarks in passing that "these equations [provide] the electromagnetic basis of the Lorentzian electrodynamics and optics of moving bodies."[17] He might, with the resources then available to him (student of Föppl that he was),

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[15] To gain a sense of those one should look into the pages of H. A. Lorentz' *The Theory of Electrons* (1909) which, though based on lectures given at Columbia University in 1906, seems in many respects thoroughly modern.

[16] "On the electrodynamics of moving bodies," (1905).

[17] We infer that Einstein had become familiar with Lorentz' *Versuch einer Theorie der elektrischen und optischen Erscheinungen in bewegten Korpern*, which had appeared in 1895.

more compactly have written

$$\begin{aligned}\tfrac{1}{c}\left\{\tfrac{\partial\mathbf{E}}{\partial t}+\mathbf{u}\rho\right\}&=\operatorname{curl}\mathbf{B}\\ -\tfrac{1}{c}\tfrac{\partial\mathbf{B}}{\partial t}&=\operatorname{curl}\mathbf{E}\\ \rho&=\operatorname{div}\mathbf{E}\\ \text{NOTE:}\quad 0&=\operatorname{div}\mathbf{B}\quad\text{is missing from Einstein's list}\end{aligned}$$

but both the structure of Einstein's subsequent argument and a feature of the result to which it leads ($\mathbf{E}_{\parallel}$ transforms differently from $\mathbf{E}_{\perp}$) made it most expedient for him to work in component form; there is, in any event, not a vector to be found in this or (so far as I have been able to discover) any of his early papers.

Here our story takes a Dickensian turn. Vector analysis—resented stepchild of the quaternion—was brought into the world by Gibbs/Heaviside to be of service to electrodynamics. But scarcely had she taken her seat at the workbench when she was informed that her services had been rendered obsolete, and that she should seek employment elsewhere. It was an unwitting Einstein who was responsible for this turn of events, though it was Minkowski who actually delivered the pink slip, for it was Minkowski who realized that Einstein's deepest accomplishment had been to weld space and time into a single metric entity: 4-dimensional spacetime. And that in such an enlarged setting both vector analysis and (somewhat surprisingly) quaternions lost the force of their former claims to special "naturalness of expression." Minkowski observed (1907) that the Maxwell-Lorentz equations—which I henceforth understand to read

$$\boldsymbol{\nabla}\cdot\mathbf{E}=\rho \tag{1.1}$$

$$\boldsymbol{\nabla}\times\mathbf{B}-\tfrac{1}{c}\tfrac{\partial}{\partial t}\mathbf{E}=\tfrac{1}{c}\mathbf{j} \tag{1.2}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B}=0 \tag{1.3}$$

$$\boldsymbol{\nabla}\times\mathbf{E}+\tfrac{1}{c}\tfrac{\partial}{\partial t}\mathbf{B}=\mathbf{0} \tag{1.4}$$

—can, from a transformation-theoretic point of view (and from other points of view as well), more usefully be written

$$\partial_\mu F^{\mu\nu}=J^\nu \tag{2.1}$$

$$\partial_\mu G^{\mu\nu}=0 \tag{2.2}$$

where $F^{\mu\nu}$, $G^{\mu\nu}$ and J^ν are the elements of

$$\mathbb{F}\equiv\begin{pmatrix}0 & -E_1 & -E_2 & -E_3\\ +E_1 & 0 & -B_3 & +B_2\\ +E_2 & +B_3 & 0 & -B_1\\ +E_3 & -B_2 & +B_1 & 0\end{pmatrix} \tag{3.1}$$

$$\mathbb{G}\equiv\begin{pmatrix}0 & +B_1 & +B_2 & +B_3\\ -B_1 & 0 & -E_3 & +E_2\\ -B_2 & +E_3 & 0 & -E_1\\ -B_3 & -E_2 & +E_1 & 0\end{pmatrix} \tag{3.2}$$

$$J \equiv \tfrac{1}{c}\begin{pmatrix} c\rho \\ j_1 \\ j_2 \\ j_3 \end{pmatrix} \tag{3.3}$$

and where the following now-standard conventions have been honored: $x^0 \equiv ct$, $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$, μ ranges on $\{0, 1, 2, 3\}$, summation on repeated indices is understood. Minkowski pointed out, moreover, that the relationship of $\mathbb{G}$ to $\mathbb{F}$ can be described

$$G^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}G_{\alpha\beta} \quad \text{with} \quad G_{\alpha\beta} \equiv \tfrac{1}{2}\epsilon_{\alpha\beta\kappa\lambda}F^{\kappa\lambda} \tag{4}$$

provided the $g_{\mu\nu}$ used to manipulate indices are taken to refer to the discovered metric structure of spacetime: $(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu$ with

$$\|g_{\mu\nu}\| \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad : \quad \text{Lorentz/Minkowski metric} \tag{5}$$

Minkowski was, *inter alia*, drawing attention to the special aptness of language and notational devices supplied by what at the time was still called "the absolute differential calculus" but is today called (the terminology is due to Einstein) "tensor analysis." The history of that subject runs parallel to—but is arguably even more ancient than—the history of vector analysis. I digress to sketch the gross outlines of that history.[18]

On 8 September 1679 Gottfried Leibniz remarked in a letter to Christian Huygens that

> *"I am still not satisfied with algebra, because it does not give the shortest methods or the most beautiful constructions in geometry. This is why I believe that, so far as geometry is concerned, we need still another analysis which is distinctly geometrical or linear and which will express* situation [situs] *directly as algebra expresses* magnitude. . . *I believe that by this method one could treat mechanics almost like geometry. . . "*

[18] My primary source in preceding pages has been Michael Crowe, *A History of Vector Analysis* (1967); this wonderful essay, which emphasizes the history of quaternions as it relates to the history of vector analysis but has much to say also about collateral developments, was republished by Dover in 1995. I have borrowed also from Thomas Hankins' *Sir William Rowan Hamilton* (1980), and from the publications of Maxwell and Lorentz already cited. In the absence (so far as I am aware) of a work comparable to Crowe but devoted to the history of tensor analysis and the exterior calculus, I have had to make do with material gleaned from E. T. Bell, *The Development of Mathematics* (1945) (especially pp. 198–211, 353–360 and 420-453) and the 2nd edition (1991) of Carl Boyer's *A History of Mathematics* (especially pp. 584–586 and 623–625).

Leibniz' thoughts in this regard were not published until 1833, and a decade later inspired the Jablonowski Gesellschaft der Wisenschaft to offer a prize for the best essay relating to the further development of Leibniz' idea. The author of the only essay submitted (and recipient of the prize) was Hermann Grassmann (1809–1877).

Hamilton and Grassmann were roughly contemporaneous (Grassmann was five years younger than Hamilton, whom he survived by twelve years), and shared a deep interest in philology (Hamilton is reported to have had some command of thirteen languages by the time he was thirteen years old, while Grassmann was the author of several Latin texts and in his later years acquired some reknown as a Sanscrit scholar). Both wrote ponderous mathematical works so encrusted with dense philosophy as to be virtually unreadable. But in the main they are a study in opposites. Hamilton was a celebrated prodigy, who while an undergraduate at Trinity College of Dublin University took every prize within sight and in 1827—not yet turned twenty-two and not yet graduated—was appointed Andrews Professor of Astronomy and Irish Astronomer Royal; Hamilton had become a great man while still a little boy. Grassmann, on the other hand, was so much the opposite of a prodigy that his father (Julius Grassmann, a teacher of mathematics and physical science in the local gymnasium) remarked that he would pleasantly surprised if his son were to achieve success even as a gardener. After several terms at the University of Berlin (where he studied theology and philology but no mathematics, except from geometry and trigonometry texts written by his father[19]) he returned to his hometown to prepare for a career similar to his father's. In the first of many unsuccessful efforts to gain a university appointment, he in 1840 completed a long essay on the theory of tides which, though it made no impression upon the examination committee, contained the elements of a rudimentary vector algebra. Impressed by the discovery that those methods permitted him to simplify many of the arguments found in Lagrange's *Méchanique analytique*, Grassmann at length undertook the intensive research effort (elaboration of work done a decade earlier) which resulted in the publication, in *1844*—almost simultaneously with Hamilton's discovery of quaternions—of the famously obscure *Ausdehnungslehre*. When the thick philosophical underbrush has been cleared away, the mathematical structure described there is found to be so vast and so general that it radiates into every remote corner of what was to become multilinear albegra; it anticipates (in addition to much else) the yet-to-be-invented theory of matrices and algebra of tensors, and yields all possible generalizations of the theory of quaternions. Remarkably and almost without precedent,[20] Grassmann's "geometrical algebra" (or "calculus of extension") assumes space to be n-dimensional. In essence, Grassmann wrote

[19] It was from the first of those that Grassmann acquired the germ of the idea of "geometrical multiplication" that was to become central to his own work.

[20] Some remarks pertaining to the possibility of an n-dimensional geometry had been published by Cayley in 1843. With that exception, all geometry at the time remained stuck in three dimensions.

$X = x^1 e_1 + x^2 e_2 + \cdots + x^n e_n$ where $\{e_1, e_2, \ldots, e_n\}$ are in effect "hypercomplex numbers," defined $X + Y$ in (what we now take to be) the natural way, and contemplated the range of meanings that might be assigned to the product $X \cdot Y$.[21] It was from *Ausdehnungslehre* that Grassmann extracted the *Die Geometrische Analyse geknüpft und die von Leibnitz erfundene geometrische Charakteristik* that in 1846 won for him the aforementioned prize... but no readers. Colleagues—Gauss, amongst many others—to whom Grassmann presented copies of his masterpiece (for so it is today regarded) professed themselves unable to penetrate beyond the first few pages, though in intuitive support of what they took to be its objectives; they gave Grassmann some encouraging pats on the back, but no offers of university appointment.

In 1862 Grassmann published, at his own expense, *Die Ausdehnungslehre: Vollständig und in strenger Form bearbeitet*, which contained much new material, material again far in advance of its time... and which again was received, at least initially, with oblivious silence. Grassmann (who was the father of eleven children) appears to have been an exceptionally vital man, productively active in a great variety of affairs both visionary and mundane, which he pursued simultaneously with his mathematics, so he might take exception to the element of "tragedy" which has attached to his memory. But it is true that only in the final years of his life was he rewarded by evidence that he would one day be read and appreciated. Bell (with characteristic overstatement) remarks that Hamilton had only one student (Tait) and that so also did Grassmann: William Clifford (who was well acquainted also with the theory of quaternions) became in England during the 1870's a vigorous champion of Grassmann's work—the theory of Clifford algebras derives from that involvement—but his influence was cut short by his death in 1879, at the age of only thirty-four.[22] Two other potential champions (Hankle and Clebsch) also died shortly after declaring their enthusiasm for Grassmann's work.

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[21] For an especially helpful synopsis of Grassmann's central idea and its relation to the mathematics of his time, see pp. 203–204 in Bell's *Development of Mathematics*.

[22] Clifford's range of interests (see the list of publications appended to the Dover edition of his posthumous *The Common Sense of the Exact Sciences*), openness to fresh ideas (non-Euclidean geometry, non-commutative algebra) and remarkably clear prevision of the future geometrization of physics—at the conclusion of "On the space-theory of matter" (1870) he writes: "I hold in fact (i) That small portions of space are in fact of a nature analogous to little hills on a surface which is on average flat; namely that the ordinary laws of geometry are not valid in them. (ii) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave. (iii) That this variation of the curvature of space is what really happens in that phenomenon which we call the *motion of matter*, whether ponderable or etherial. (iv) That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity..."—recall the mind of Riemann and anticipate that of Poincaré.

Grassmann anticipated much, but influenced little; it was his fate to be appreciated only retrospectively, after other mathematicians had—in bits and pieces, and after a lapse of sometimes many decades—reproduced his train of thought.[23] While *Ausdehnungslehre* might in principle have influenced the development of vector analysis, in fact it did not, though Gibbs did have some knowledge of the work, and was destined to play a role (together with Felix Klein) in bringing about the publication of Grassmann's collected works.[24] Nor is any reference to Grassmann to be found (so far as I have been able to discover) in any of the tensor analytic literature to which Minkowski had access. . . though it is curious fact that neither in "Die Grundgleichungen für die electromagnetischen Vorgänge in bewegten Körpern" (1907) nor in the posthumous companion publication which was prepared for publication by Max Born (1911) did Minkowski acknowledge—or even give indication that he was aware of the fact—that he was making use of tensor analytic methods;[25] he does remark (in a footnote) that his methods have much in common with "Cayley's matrix calculus," which (he supposed to be) derived from "Hamilton's

[23] The remark pertains even to some of Grassmann's work in other spheres; his "Neue Theorie der Elektrodynamik" (1845) attracted no notice until some of the results reported there were reproduced by Rudolph Clausius in the 1870's. Maxwell does make reference to Grassmann's paper, but there is no evidence that he was familiar with *Ausdehnungslehre*, to which Grassmann refers, and from which he borrows his methodology.

[24] Klein—who once remarked that he felt himself to be more intimately related to Clifford than to any other geometer, and who (like Clifford?) was introduced to Grassmann's work by study of Hankel's *Theorie der complexen Zahlensysteme* (1867)—is one of the few mathematicians ever to declare a direct indebtedness to Grassmann. In 1911 he wrote to the editor of Grassmann's collected works that "As is well known, Grassmann in his *Ausdehnungslehre* is an affine, rather than a projective, geometer. This became clear to me in the late fall of *1871* and (besides the study of Möbius and Hamilton. . .) led to my conception of my later Erlanger Program." Klein was only four years younger than Clifford, and only twenty-three when, in 1872, he described the principles which in his view should inform all geometrical research. The influence of Grassmann is strongly evident also in Klein's *Elementary Mathematics from an Advanced Standpoint* (1908). The allusion to Möbius is an allusion to his *Der barycentrische Calcul* (1827), with which Grassmann had familiarized himself immediately prior to the work that produced *Ausdehnungslehre*. While Möbius' monograph merits mention in any comprehensive account of the history of linear algebra, it is more often recalled as a landmark in the early history of topology; it is where the concept of "orientability" comes from. And the Möbius strip. And, by implication, the Klein bottle.

[25] This circumstance may account for the fact that Einstein had to wait until 1912 to learn from Marcel Grossmann of the potential relevance of the "Ricci calculus" to his general relativistic effort; see Chapter 12 of A. Pais' *'Subtle is the Lord. . . ': The Science and Life of Albert Einstein*(1982) for illuminating discussion.

quaternion calculus."[26] But Minkowski seems not to have expected his intended readers (mathematical physicists) to know what a matrix is, and to have been content to invent his matrix theory and *de facto* tensor analysis (which was, after all, pretty rudimentary: he was working with tensors of low rank, and concerned only with globally linear transformations in flat space) as he went along.

The tensor analysis to which Minkowski did not allude, and of which he may or may not have possessed a command, is a creation mainly of Gregorio Ricci (1853–1925) and his former student, Tuillio Levi-Civita (1873–1941); it is the culminaton of an effort which began in the mid-1880's and achieved definitive form in the jointly written reviews which were published in several languages in about 1900. It builds, of course, upon the differential geometry created by Gauss and Riemann, and elaborated by (among others) Elwin Christoffel (1869), but drew motivation also from the theory of algebraic invariants which was actively pursued during the mid-Nineteenth Century by Cayley, Sylvester and many other mathematicians.[27] It was—then as now—widely held that the tensor calculus was destined to sterile obscurity until rescued from that oblivion by Einstein; whatever the justice of that assessment (to which I would take exception), it is true that the long section "B. Mathematical Aids to the Formulation of Generally Covariant Equations" in his "Die Grundlage der allgemeinen Relativitätstheorie" (1916) does read like an introductory tensor text, and that the excitement inspired by Einstein's accomplishment did stimulate interest in Ricci's.

Élie Cartan (1869–1951) began his long research career where Sophus Lie (1842–1899) left off; i.e., with the theory of finite-dimensional Lie groups, which in his celebrated thesis (1894) he brought to a kind of closure. Five years later, after he and Hermann Weyl had achieved a complete classification of the hypercomplex number systems (Lie algebras) to which such groups give rise, Cartan turned his attention to the topological/global properties of Lie groups and associated differential geometry, and it was as a by-product of this effort that the exterior calculus came into being. Building upon earlier work by Poincaré and Goursat, Cartan sought to bring into sharper focus the previously neglected *integral* aspects of the Ricci calculus, and at the same time to remove what he considered to be a characteristic defect of that formalism; he writes

[26] Arthur Cayley is usually given credit for the invention, in 1858, of the theory of matrices. He himself insisted that he took not quaternions but the theory of determinants—the history of which can be traced back to work done in 1772 by Laplace and Vandermonde, but in its modern form is a creation mainly of Cauchy (1812)—and the transformational properties of systems of linear equations as his point of departure.

[27] For a good account of these complicated cross-currents, see Chapter 20 in Bell's *Development of Mathematics*.

> *". . . [these objectives] required ideas from tensor analysis. I have tried to do this while emphasizing the essential geometrical features and keeping in close contact with Euclidean geometry. The utility of the absolute differential calculus of Ricci and Levi-Civita must be tempered by an avoidance of excessively formal calculations, where the debauch of indices disguises an often very simple geometric reality. It is this reality that I have sought to reveal. I present an account of the interesting problems of those spaces which, while locally Euclidean, are from the point of view of 'analysis situs,' different from our ordinary space. This involves the 'Clifford-Klein space forms' of the German school of geometry. . . "*[28]

The "absolute differential calculus of Ricci and Levi-Civita" (tensor calculus) rests upon the foundation provided by tensor algebra. Cartan's creation—the "exterior calculus"—rest similarly upon the "exterior algebra" which results when tensor algebra is subjected to certain specializing restrictions of which antisymmetry is the hallmark. Modern authors[29] consider the terms "exterior algebra" and "Grassmann algebra" to be synonymous, but Cartan (beyond his allusion to ". . . the German school of geometry. . . ") acknowledges no special indebtedness to Grassmann. One has, however, only to compare Chapter 1 and the beginning of Chapter 8 in Cartan's *Geometry of Riemannian Spaces* with "Chapter II: The Grassmann Determinant Principle for the Plane" and "Chapter III: The Grassmann Principle for Space" in the second volume (the geometrical volume) of Klein's *Elementary Mathematics from an Advanced Standpoint* to see how enormous Grassmann's influence upon Cartan—and

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[28] The quotation is from the preface to the English translation (1983) of the 2nd edition (1951) of Cartan's *Leçons sur la Géométrie des Espaces de Riemann*, which first appeared in 1928. There is irony in the fact that the "debauch d'indices" lamented by Cartan was carried to its highest degree of perfection by one of Cartan's own students; J. A. Schouten's *Ricci-Calculus* (1923, revised in 1954) is virtually unreadable. And it is salutary to be reminded that a mathematician so eminent as Hermann Weyl considered that a few indices might, on balance, be a good thing; at the end of §6 in his *Raum, Zeit, Materie* (1923) he writes ". . . Various attempts have been made to set up a standard terminology in this branch of mathematics involving only the [tensors] themselves and not their components, analogous to that of vectors in vector analysis. This is highly expedient in the latter, but very cumbersome for the much more complicated framework of the tensor calculus. In trying to avoid continual reference to components we are obliged to adopt an endless profusion of names and symbols in addition to an intricate set of rules for carrying out calculations, so that the balance of advantage is considerably on the negative side. An emphatic protest must be entered against these orgies of formalism which are threatening the peace of even the technical scientist."

[29] See, for example, R. Abraham, J. E. Marsden & T. Ratiu, *Manifolds, Tensor Analysis, and Applications* (2nd edition, 1988), p. 397; or §256.O in the *Encyclopedic Dictionary of Mathematics* (1993)

Klein[30]—actually was.

Fundamental to work in this area for now more than a century and a half has been the determinant, which is the source (or at least the repository) of all the antisymmetry. It was Cayley's observation (1843) that statements of the form

$$\text{area of triangle with specified vertices} = \tfrac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$\text{equation of line through specified points:}\quad \begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} = 0$$

generalize straightforwardly that first alerted him to the possibility of an n-dimensional geometry. Grassmann, independently pursuing the same train of thought, observed moreover that

$$\begin{aligned} x_1y_2 - x_2y_1 &= \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix}_{\text{eliminate } 1^{\text{st}} \text{ column}} \\ y_2 - y_1 &= \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix}_{\text{eliminate } 2^{\text{nd}} \text{ column}} \\ x_2 - x_1 &= \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix}_{\text{eliminate } 3^{\text{rd}} \text{ column}} \end{aligned}$$

also admit of geometrical interpretation, and also generalize straightforwardly, and was led thus to what Klein calls "Grassmann's principle." These remarks supply all the material needed to construct a theory of multiple integration, and it was in part to render most natural the expression of that theory that Cartan devised the exterior calculus; he writes (at the beginning of §181 in a work already cited) "we refer to differential forms with exterior multiplication, or more briefly, exterior differential forms, as those forms which occur... in multiple integrals; these obey certain rules..." which he illustrates by appeal to several variants of Stokes' theorem.[31]

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[30] Grassmann and Möbius are the two most frequently cited authors in Klein's geometrical volume, and the influence especially of Grassmann can be detected on virtually every page of a work which remains as elegantly lucid today as Grassmann's own work remains obscure. In his first volume, which treats algebraic topics, Klein draws appreciative attention also to Grassmann's work relating to the construction of number systems (*Lehrbuch der Arithmetik*, 1861), which he compares favorably to Giuseppe Peano's *Arithmetices principia nova methodo exposita* (1889) and *Formulaire de Mathématiques* (1892–1899).

[31] Stokes theorem, which was known to Kelvin already by 1850, made its first public appearance in the Cambridge tripos examination of 1854. Among those who sat for the exam was Maxwell, whose performance earned him the distinction of Second Wrangler.

The exterior calculus was in place by about 1920, and at some undiscovered later date somebody[32] noticed that the Maxwell-Lorentz equations (1) can, in notations standard to the exterior calculus, be written

$$*\mathbf{d}*\mathbf{F} = \mathbf{J} \qquad \text{and} \qquad *\mathbf{d}\ \ \mathbf{F} = \mathbf{0} \tag{6}$$

The adjustments—notational, most obviously, but also physical/conceptual—which distinguish (6) from the equations originally put forward by Maxwell could hardly be more striking. The story of how we got here from there, even when told in such bald outline as I have done, is marked by an almost operatic variety of human emotions, ranging from table-pounding insistence to the despair of misunderstood neglect (sometimes, as in the case of Heaviside, simultaneous). And it illustrates well the ancient truism that what we think—what we *can* think—is shaped and limited by the language in which we think, and that, conversely, the force of insistent thought, however vague, can in the end reshape language.[33] All participants (Tait, Gibbs, others) in the "struggle for existence"[34] which raged (in the pages of *Nature* and elsewhere) during the 1890's accepted the validity of that truism; their passionate disagreement (if we set aside questions of priority and similar distractions) had to do with its concrete implications in the particular instance, and clarity with regard to those tends to be achieved only in retrospect, after a sufficient body of evidence has accumulated to permit a response to Kelvin's question: "Does the proposed new language enable one to do what otherwise could not be done?"[35]

The events to which I have alluded—which, though of a mathematical nature, were to a remarkable degree sparked by electrodynamical developments—span an interval of seventy-five years, from beginning to end. Another such interval has elapsed since equations (6) were (or might in principle have been) first written down but—amazingly to me—they have yet to find their way into

[32] I have yet do discover an author willing to cite his sources in this regard. The earliest reference known to me is R. Debever, *Espaces de l'ectromagnétisme* (Colloque de géométrie différentielle, Louvain 1951), but I strongly suspect that equations (6) were first written down decades before that.

[33] A story often told in connection with Gibbs—a physicist, it is relevant to remember—comes to mind. It was reportedly his habit to sit in silence through meetings of the Yale faculty. But one one occasion (the debate, as I heard the story, had to do with the status of the language requirement) he was provoked to rise to his feet and inform those assembled that "Mathematics is a language." Paul Samuelson, who was proud to claim direct descent (by way of Wilson) from Gibbs, appended those words as a kind of subtitle to the first edition of his *Foundatons of Economic Analysis*.

[34] I take the phrase from the title of Chapter 6 in Crowe's *A History of Vector Analysis*, cited previously.

[35] Quaternions, when applied to the physics of a century ago, ultimately failed that test. How they fair when applied to more recent physics is still being debated; see Stephen L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields* (1995) and sources cited there.

the textbooks read by physicists. They do (typically in the somewhat garbled form one expects of authors who are uncomfortable with the physics) appear in some mathematical texts, where they are pressed into service to illustrate the potential utility of the formalism, but those demonstrations lack force because they fail Kelvin's test: they achieve no result which cannot be achieved as simply by more familiar means.

It is my impression that contemporary mathematical physicists who do possess a command of the exterior calculus tend nevertheless to avoid use of that formalism in their publications for the same sound reason that Newton (whose reasons were, however, seldom simple) avoided reference to the calculus in the *Principia*, the same reason that Maxwell[36] and the quaternionists of a century ago avoided quaternions in their physical writing: fear that they will not be understood by their intended readers.

It is my strong feeling that it is time—long past time—to bring that unfortunate state of affairs to an end. It is wrong to suppose that one need be a state-of-the-art differential geometer to make effective use of the exterior calculus. The formalism imposes demands certainly not greater than those of tensor analysis (from which it arises as a powerful special case), and in practical applications does its work even more simply and swiftly than vector analysis.[37] And, as it is my intention here to demonstrate, it does pass Kelvin's test, does put us in position to say new things about some moderately interesting new questions, to say things that are quite beyond the reach of vector analysis, and which if formulated in the language of tensor analysis would entail such a "debouch d'indices" as to remain almost certainly unsaid.

1. Present objectives. In a recent essay[38] I chanced to notice that the equations which serve to provide an exterior formulation of the Maxwell-Lorentz equations serve also to support a population of "theories of Maxwellian design"—a formal population of theories, that is to say, which share with physical electrodynamics the properties that

$$\ast\mathbf{d}\ast\mathbf{F} = \mathbf{J} \tag{7.1}$$

$$\ast\mathbf{d}\ \ \mathbf{F} = \mathbf{0} \tag{7.2}$$

$$\text{rank}(\mathbf{F}) = \text{rank}(\ast\mathbf{F}) \tag{7.3}$$

The latter condition—since in point of mathematical fact

$$\text{rank}(\ast\mathbf{F}) = \text{dimension} - \text{rank}(\mathbf{F})$$

—carries with it the implication that such a theory becomes possible if and only if the dimension of spacetime is even

$$\text{dimension} = 2 \cdot \text{rank}(\mathbf{F})$$

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[36] See again the quotation on p. 4.

[37] Compare its basic statements $\mathbf{dd} = 0$ and $\ast\ast \sim 1$ with the long list of familiar but quite unmemorable identities reproduced inside the covers of (for example) David Griffiths' *Introduction to Electrodynamics* (1989).

[38] "Electrodynamical applications of the exterior calculus" (1996).

and can by natural associations be interpreted to mean that

$$\text{number of "electric" components of } \mathbf{F} = \text{number of "magnetic" components}$$

It was found that in all essential respects the formal properties of the theories here in question mimic properties of the

$$\text{physical case} \quad : \quad \text{dimension} = 4$$

In a subsequent essay[39] I looked in deep detail to the simplest instance of such a theory (dimension = 2), which, though "simple," was found to be far from trivial, and far from powerless to teach us useful things about the real world. More immediately to the point, it was found that 2-dimensional gauge theory gives rise to an "electrodynamics" which is *distinct* from the theory to which I have just alluded—a theory in which (7.1) and (7.2) still pertain

$$\ast\mathbf{d}\ast\mathbf{F} = \mathbf{J} \tag{8.1}$$

$$\ast\mathbf{d}\ \ \mathbf{F} = \mathbf{0} \tag{8.2}$$

but in which (7.3) must, *except in the case dimension* = 4, be abandoned. It was brought thus to my attention that the gauge-theoretic mechanism works (in the sense that it yields "theories of Maxwellian design") quite generally; it yields systems of type (8) with the property that

$$\text{rank}(\mathbf{F}) = 2 \text{ irrespective of dimension} \geq 2$$

I had in hand, by this point, *two distinct populations* of dimensional generalizations of Maxwellian electrodynamics, populations which *intersect in the physical case* (dimension = 4), and which are, in at least this sense, "equally plausible." The situation is illustrated in the FIGURE 1. My objective in the present essay is, taking (8) to comprise the *definition* of a "theory of Maxwellian design," to explore the possibility of "filling in the dots." To pursue that objective we need, first of all, to acquire a command of the

2. Basic rudiments of the exterior calculus. I will attempt to demonstrate how far one can go with how little, how swift and essentially elementary arguments based upon the exterior calculus can be, at least in favorable cases. Readers made apprehensive by such a procedure may want to consult (or at least to skim) the literature before proceeding.[40]

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[39] "'Electrodynamics' in 2-dimensional spacetime" (1997).

[40] The primary references are, of course, to Cartan, who appears to have spent his last years overseeing the publication of work done decades before; his *Leçons sur la théorie des spineurs*, based on work completed in 1913, was published only in 1937 (and in English translation in 1981), his *Systèmes différentiels*

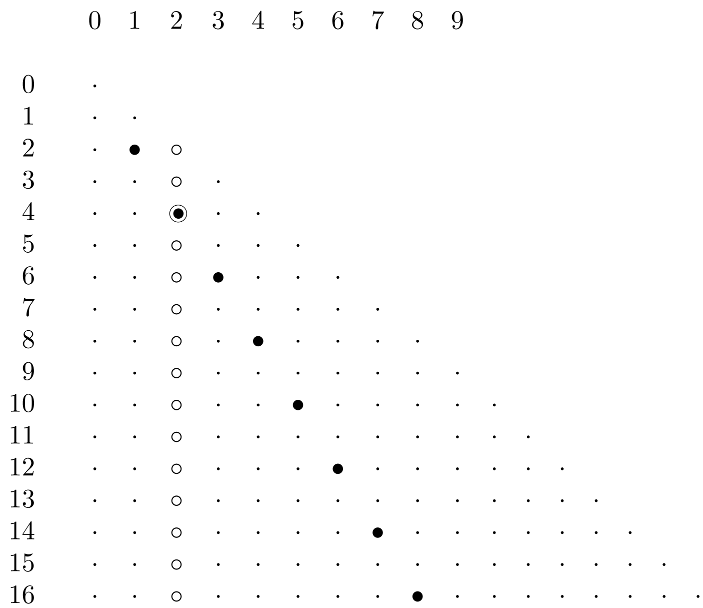

FIGURE 1: *Rank runs →, dimension runs ↓. The black dots ● mark the placement of the theories defined by (7), open circles ○ mark the placement of the gauge-theoretic realizations of (8). The populations intersect at the physical case marked ◉*

We borrow first from the exterior calculus some language: when we assert that "**F** is an n-dimensional p-form" we mean simply that **F** is an n-dimensional

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extérieurs et leurs applications géométriques appeared in the year of his death (1951), and his *Leçons sur les invariants integraux* (1921) was reprinted posthumously (1958). Henri Cartan (Élie's eldest son, born in 1904, and a founding member of Bourbaki) published *Formes différentielles* in 1970. These days, most English-speaking physicists learn their exterior calculus from Flanders' *Differential Forms, with Applications to the Physical Sciences* (1963) or perhaps from Misner, Thorne & Wheeler's "Chapter 4: Electromagnetism and Differential Forms" in *Gravitation* (1970). In addition to references cited there, I would emphasize the utility of the monograph by Abraham, Marsden & Ratiu (mentioned previously) and the wonderful text *Advanced Calculus* by R. C. Buck (1956). Some readers may find value also in H. M. Edwards, *Advanced Calculus: A Differential Forms Approach* (1969, reprinted in 1994), for which Buck wrote the enthusiastic Introduction. It is in the presumption that my reader will have looked into some of those works that I will allow myself the convenience of referring most frequently to my own work in this field. Here the sources are ELECTRODYNAMICS (1972) and the two essays (1996 & 1997) mentioned previously.

totally antisymmetric tensor of rank p. The (non-standard) notation

$$\mathbf{F} \prec F^{i_1 i_2 \cdots i_p}$$

will be read "$\mathbf{F}$ is a tensor whose elements are given by $F^{i_1 i_2 \cdots i_p}$". Total antisymmetry entails, of course, that $0 \leq p \leq n$.

We assume spacetime to be endowed with metric structure g_{ij}, and will use g_{ij} to raise and lower indices at will, after the manner standard to tensor analysis. That's mere mathematics; physically, we will, until further notice, assume the metric to possess the Lorentzian structure implicit in (5)

The action of the exterior differentiation operator $\mathbf{d}$ is understood to entail ordinary partial differentiation *followed by antisymmetrization*:

$$\mathbf{dF} \prec \tfrac{1}{p!}\delta_{i_1 i_2 \cdots i_{p+1}}{}^{b a_1 \cdots a_p}\partial_b F_{a_1 \cdots a_p} \tag{9}$$

Use has been made here of the "generalized Kronecker delta," which can be described[41]

$$\delta^{i_1 i_2 \cdots i_p}{}_{j_1 j_2 \cdots j_p} = \begin{vmatrix} \delta^{i_1}{}_{j_1} & \delta^{i_1}{}_{j_2} & \cdots & \delta^{i_1}{}_{j_p} \\ \delta^{i_2}{}_{j_1} & \delta^{i_2}{}_{j_2} & \cdots & \delta^{i_2}{}_{j_p} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{i_p}{}_{j_1} & \delta^{i_p}{}_{j_2} & \cdots & \delta^{i_p}{}_{j_p} \end{vmatrix} \tag{10}$$

and serves to "mechanize" the required sum over signed permutations. While it is generally the case that application of $\mathbf{d}$ achieves

$$\mathbf{d} \quad : \quad p\text{-form} \longrightarrow (p+1)\text{-form} \tag{11}$$

it is important to notice that (because it is meaningless to speak of "p-forms with $p > n$")

$$\mathbf{d} \text{ annihilates } n\text{-forms} \tag{12}$$

and that in consequence of $\partial_i \partial_j = \partial_j \partial_i$ we have the "Poincaré lemma"

$$\mathbf{dd} \text{ annihilates } p\text{-forms} \quad : \quad \text{all } p \tag{13}$$

We will make important use of the so-called "converse of the Poincaré lemma," which asserts that if $\mathbf{A}$ is a p-form ($p \geq 1$) such that $\mathbf{dA} = \mathbf{0}$, then there exists a $(p-1)$-form $\mathbf{B}$ such that $\mathbf{A} = \mathbf{dB}$. $\mathbf{B}$ is, moreover, non-unique; it is determined only up to a "gauge transformation"

$$\mathbf{B} \longrightarrow \mathbf{B}' = \mathbf{B} + \mathbf{dC} \qquad \text{where } \mathbf{C} \text{ is an arbitrary } (p-2)\text{-form} \tag{14}$$

But I do not anticipate need of the amazing formula that permits one (under weak hypotheses, and up to gauge) actually to *exhibit* such a $\mathbf{B}$.[42]

[41] See §3 in my 1996 essay.
[42] See §1 in my 1997 essay. Such a formula, by its mere existence, supplies the constructive proof of the theorem in question.

The mixed tensor $\delta^i{}_j$ possesses the remarkable property that the defining statement

$$\delta^i{}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

is, if valid by prescription on one coordinate system, automatically valid in *all* coordinate systems. "Universality" in this same sense attaches also to the Levi-Civita tensors

$$\epsilon_{i_1\cdots i_n} = \begin{cases} +1 & \text{if } i_1 i_2 \cdots i_n \text{ is an even permutation of } 12\cdots n \\ -1 & \text{if } i_1 i_2 \cdots i_n \text{ is an odd permutation of } 12\cdots n \\ \;\;0 & \text{otherwise; i.e., if any index is repeated} \end{cases} \tag{15.1}$$

$$\ni^{i_1\cdots i_n} = \begin{cases} +1 & \text{if } i_1 i_2 \cdots i_n \text{ is an even permutation of } 12\cdots n \\ -1 & \text{if } i_1 i_2 \cdots i_n \text{ is an odd permutation of } 12\cdots n \\ \;\;0 & \text{otherwise; i.e., if any index is repeated} \end{cases} \tag{15.2}$$

provided $\epsilon_{i_1\cdots i_n}$ is assumed to transform as a tensor *density of weight minus one*, and $\ni^{i_1\cdots i_n}$ as a *density of weight plus one*. The previously encountered "generalized Kronecker delta" comes into being as a natural artifact of the objects thus defined:

$$\ni^{i_1 i_2\cdots i_n}\,\epsilon_{j_1 j_2\cdots j_n} = \delta^{i_1 i_2\cdots i_n}{}_{j_1 j_2\cdots j_n} \tag{16}$$

And by contraction (the ghost of Grassmann haunts the following constructions) one obtains

$$\left.\begin{aligned} \delta^{i_1 i_2\cdots i_{n-1}a}{}_{j_1 j_2\cdots j_{n-1}a} &= \delta^{i_1 i_2\cdots i_{n-1}}{}_{j_1 j_2\cdots j_{n-1}} \\ \delta^{i_1 i_2\cdots i_{n-2}a_1 a_2}{}_{j_1 j_2\cdots j_{n-2}a_1 a_2} &= 2\cdot\delta^{i_1 i_2\cdots i_{n-2}}{}_{j_1 j_2\cdots j_{n-2}} \\ \delta^{i_1 i_2\cdots i_{n-3}a_1 a_2 a_3}{}_{j_1 j_2\cdots j_{n-3}a_1 a_2 a_3} &= 3\cdot 2\cdot\delta^{i_1 i_2\cdots i_{n-3}}{}_{j_1 j_2\cdots j_{n-3}} \\ &\;\vdots \\ \delta^{i a_1\cdots a_{n-1}}{}_{j a_1\ldots a_{n-1}} &= (n-1)!\cdot\delta^i{}_j \\ \delta^{a_1 a_2\cdots a_n}{}_{a_1 a_2\cdots a_n} &= n! \end{aligned}\right\} \tag{17}$$

These objects—which (see again (10)) can be described as determinants of diminishing size, with simple δ-elements—acquire both their universality and their tensorial weightlessness "by inheritance," from properties of $\epsilon_{i_1 i_2\cdots i_n}$ and its companion.

Even in the absence of a metric, one can with the aid of these universally available devices define a (duplex) "Levi-Civita dualization process"

$$\left.\begin{aligned} F^{i_1\cdots i_p} &\longrightarrow F^{\text{dual}}_{j_1\cdots j_{n-p}} \equiv \tfrac{1}{p!}\,\epsilon_{j_1\cdots j_{n-p}a_1\cdots a_p}F^{a_1\cdots a_p} \\ F_{i_1\cdots i_p} &\longrightarrow F_{\text{dual}}^{j_1\cdots j_{n-p}} \equiv \tfrac{1}{p!}\,\ni^{j_1\cdots j_{n-p}a_1\cdots a_p}F_{a_1\cdots a_p} \end{aligned}\right\} \tag{18}$$

which is "self-inversive" in this sense:

$$
\begin{aligned}
F^{i_1\cdots i_p} \rightarrow F^{\text{dual}}_{j_1\cdots j_{n-p}} \rightarrow F^{i_1\cdots i_p}_{\text{dual dual}} &= \tfrac{1}{(n-p)!p!}\,\ni^{i_1\cdots i_p b_1\cdots b_{n-p}}\ \epsilon_{b_1\cdots b_{n-p}a_1\cdots a_p}F^{a_1\cdots a_p} \\
&= (-)^{p(n-p)}\tfrac{1}{(n-p)!p!}\,\ni^{i_1\cdots i_p b_1\cdots b_{n-p}}\ \epsilon_{a_1\cdots a_p b_1\cdots b_{n-p}}F^{a_1\cdots a_p} \\
&= (-)^{p(n-p)}\tfrac{1}{p!}\delta^{i_1\cdots i_p}{}_{a_1\cdots a_p}F^{a_1\cdots a_p} \\
&= (-)^{p(n-p)}F^{i_1\cdots i_p} \\
&\sim F^{i_1\cdots i_p} \qquad (19)
\end{aligned}
$$

The prefactor

$$
(-)^{p(n-p)} = \begin{cases} -1 & \text{if } n \text{ is even and } p \text{ is odd} \\ +1 & \text{otherwise} \end{cases} \qquad (20)
$$

is an artifact of "getting the ducks lined up before we shoot 'em," and is one of two small fuss points characteristic of this subject. The other has to do with the fact that Levi-Civita dualization *tips the weight* of the tensors upon which it acts; specifically, if

$$\mathbf{F} \prec \text{ contravariant tensor density of weight } W$$

then

$$\mathbf{F}^{\text{dual}} \prec \text{ covariant tensor density of weight } W-1$$

while if

$$\mathbf{F} \prec \text{ covariant tensor density of weight } W$$

then

$$\mathbf{F}_{\text{dual}} \prec \text{ contravariant tensor density of weight } W+1$$

This last remark acquires pertinence from the circumstance that tensors A and B can be added, subtracted or set equal if and only if they have the same rank *and weight*.[43]

But if (as in the applications here contemplated) one does have access to a metric g_{ij} then the work formerly assigned to $\ni^{i_1\cdots i_n}$ can be reassigned to

$$\epsilon^{i_1\cdots i_n} \equiv g^{i_1 i_1}\cdots g^{i_n j_n}\epsilon_{j_1\cdots j_n} = g^{-1}\cdot \ni^{i_1\cdots i_n} \qquad (21)$$

Weight consistency in the preceding equation follows from the observation that

$$g \equiv \det\|g_{ij}\| \text{ transforms as a scalar density of weight } W = 2 \qquad (22)$$

and in the light of that fact it becomes natural to introduce the "Hodge star operator" $*$, the action of which can be described

$$\mathbf{F} \longrightarrow *\mathbf{F} \prec \tfrac{1}{p!}\sqrt{g}\,g^{i_1 j_1}\cdots g^{i_{n-p}j_{n-p}}\epsilon_{j_1\cdots j_{n-p}a_1\cdots a_p}F^{a_1\cdots a_p} \qquad (23)$$

---

[43] Weight considerations are, however, rendered moot if all elements of the transformation group are *unimodular*. This would, for example, become the case if in relativistic theory improper Lorentz transformations were disallowed.

The operator $*$ achieves

$$* \quad : \quad p\text{-form} \longrightarrow (n-p)\text{-form} \tag{24}$$

and, by virtue of the $\sqrt{g}$-factor, does preserve weight; it becomes therefore feasible to contemplate conditions under which $\mathbf{F}$ becomes "(anti)self-dual," which we will have physical reason to do.

Inclusion of the $\sqrt{g}$-factor entails, however, its own kind of cost: if $\mathbf{F}$ is real and $g < 0$ then $*\mathbf{F}$ is *imaginary*, and imaginary numbers (at least those of such an origin) may constitute an unwelcome intrusion into the physics at hand.[44] To sidestep the problem one might in (23) make the substitution

$$g \longmapsto |g| = \omega g \qquad \text{with } \omega \equiv \pm 1 \text{ according as } g \gtrless 0$$

But such modification of the definition of $*$ entails modification also of its properties—in place of

$$**\mathbf{F} = (-)^{p(n-p)}\mathbf{F} = \begin{cases} -\mathbf{F} & \text{if } n \text{ is even and } p \text{ is odd} \\ +\mathbf{F} & \text{otherwise} \end{cases} \tag{25}$$

one obtains the relatively more fussy statement

$$**\mathbf{F} = \omega \cdot (-)^{p(n-p)}\mathbf{F}$$

which in relativistic applications[45] becomes

$$= \begin{cases} -\mathbf{F} & \text{if } n \text{ and } p \text{ are both even} \\ +\mathbf{F} & \text{otherwise} \end{cases}$$

—and it is not, on balance, clear that one comes out ahead. My policy will to be embrace the standard definition (23) except when I have compelling reason to do otherwise, and in circumstances of the latter sort to write $\star$ in place of $*$.

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[44] We observe in this connection that if g_{ij} is (in the sense made obvious by generalization of (5)) "Lorentzian," then

$$g = (-)^{\text{space dimension}}$$

gives $g < 0$ if space is odd-dimensional; i.e., if spacetime is even-dimensional.

[45] We use the fact that in n-dimensional spacetime $\omega = (-)^{n-1}$.